Rossmoyne Senior High School

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4
Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work:	ten minutes
Working time:	one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items:	pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters
Special items:	drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

Three planes have the following equations, where a and b are constants.

$$x + z = 3$$
$$2x - y + 3z = 9$$
$$2x + y + az = b$$

(a) Determine the coordinates of the point of intersection of the three planes when $a = -3$ and $b = 3$. (2 marks)

Solution
R2+R3: $4x = 12 \Rightarrow x = 3, z = 0, y = -3$ At $(3, -3, 0)$
Specific behaviours
✓ indicates elimination of variables ✓ point of intersection (may use CAS)

(b) Determine any restrictions on the constants a and b if the planes

(i) intersect in a straight line. (3 marks)

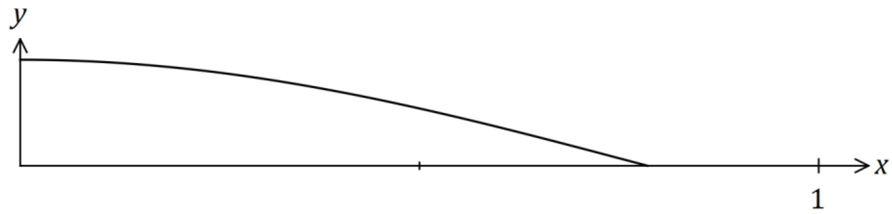
Solution
R2+R3-4R1: $(a-1)z = b - 3$ Require $a - 1 = 0$ and $b - 3 = 0$ $a = 1, \quad b = 3$
Specific behaviours
✓ eliminates two variables ✓ indicates coefficient of z and constant must both be zero ✓ states restrictions

(ii) neither intersect at a point nor in a straight line. (1 mark)

Solution
$a = 1, \quad b \neq 3$
Specific behaviours
✓ states restrictions

Question 10 **(5 marks)**

The graph of $y = 8\cos 2x$ is shown below.

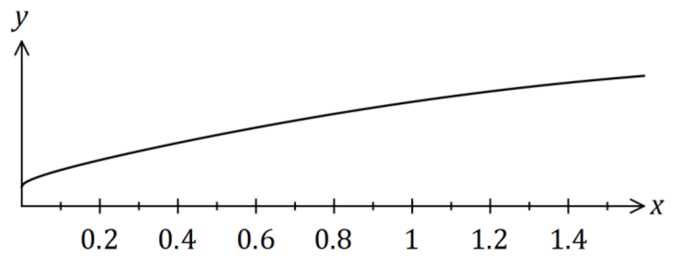


Show that when the part of the curve between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x axis, the volume of the solid generated is $8\pi^2$. Clearly indicate all trigonometric identities used.

Solution
$V = \pi \int_0^{\frac{\pi}{4}} 8^2 \cos^2 2x \, dx$ $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$ $64\cos^2 2x = 32(1 + \cos 4x)$ $= 32 + 32\cos 4x$ $V = \pi \int_0^{\frac{\pi}{4}} 32 + 32\cos 4x \, dx$ $= \pi[32x + 8\sin 4x]_0^{\frac{\pi}{4}}$ $= \pi(8\pi + 0) - \pi(0 - 0)$ $= 8\pi^2$
Specific behaviours
✓ writes correct integral to evaluate ✓ shows identities for $\cos^2 2x$ ✓ substitutes to obtain expression in terms of $\cos 4x$ ✓ integrates correctly ✓ substitutes all limits (and simplifies)

Question 11 **(5 marks)**

Part of the graph of $y = e^{2\sin\sqrt{x}}$ is shown below.



(a) Use numerical integration with three equal width trapeziums to estimate the area between the curve, the x-axis, the y-axis and the line $x = 1.5$. (4 marks)

Solution
$f(0) = 1$ $f(0.5) = 3.667$ $f(1) = 5.381$ $f(1.5) = 6.563$ $T(1) = \frac{1 + 3.667}{2} \times \frac{1}{2} = 1.167$ $T(2) = 2.262$ $T(3) = 2.986$ $\text{Area} = 6.415 \quad (3\text{ sf})$
Specific behaviours
✓ calculates $f(x)$ for $x = 0, 0.5, 1, 1.5$ ✓ uses trapezium rule ✓ area of one trapezium correct to at least 3rd decimal place ✓ area rounded to 3 or more (ie 4, 5, ...) sf

(b) Briefly explain whether your estimate is too small or large. (1 mark)

Solution
Too small, as curve is concave down.
Specific behaviours
✓ indicates too small

Question 12 **(8 marks)**

The position of particle P at any time s seconds is given by $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + 12\mathbf{k} + s(-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$, where distances are in metres.

(a) Show that when $s = 3$, P is 7 m from the origin. (2 marks)

Solution
$\mathbf{r}(3) = (-2, 3, 6)$ $\sqrt{(-2)^2 + 3^2 + 6^2} = \sqrt{49} = 7$
Specific behaviours
✓ indicates position of P ✓ shows magnitude calculation

When $s = 0$, particle Q leaves the point $(9, 13, 0)$ and moves with constant velocity, passing through the point $(6, 11, 1)$ one second later.

(b) Describe the path of Q as a vector function of time t seconds. (2 marks)

Solution
$(6, 11, 1) - (9, 13, 0) = (-3, -2, 1)$ $\mathbf{r} = 9\mathbf{i} + 13\mathbf{j} + t(-3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
Specific behaviours
✓ indicates direction vector ✓ states equation of path

(c) Determine where the paths of P and Q cross, and explain whether the particles meet. (4 marks)

Solution
$1 - s = 9 - 3t$ $-3 + 2s = 13 - 2t$ $12 - 2s = t$ Soln: $s = 4, t = 4$ Paths intersect at $(-3, 5, 4)$ Particles meet as paths cross at same time
Specific behaviours
✓ indicates equations to solve ✓ solves equations ✓ determines point ✓ states meet, with reason

Question 13 **(6 marks)**

When used in a torch, the lifetime of a single AAA battery was observed to be normally distributed with a mean of μ hours and a standard deviation of σ hours.

A student bought 40 boxes of these batteries, with 48 batteries in each box, and calculated the average lifetime for the batteries in each box. The mean of the averages was 8.31 hours and the standard deviation of the averages was 0.05 hours.

(a) Use this information to determine estimates for μ and σ. (3 marks)

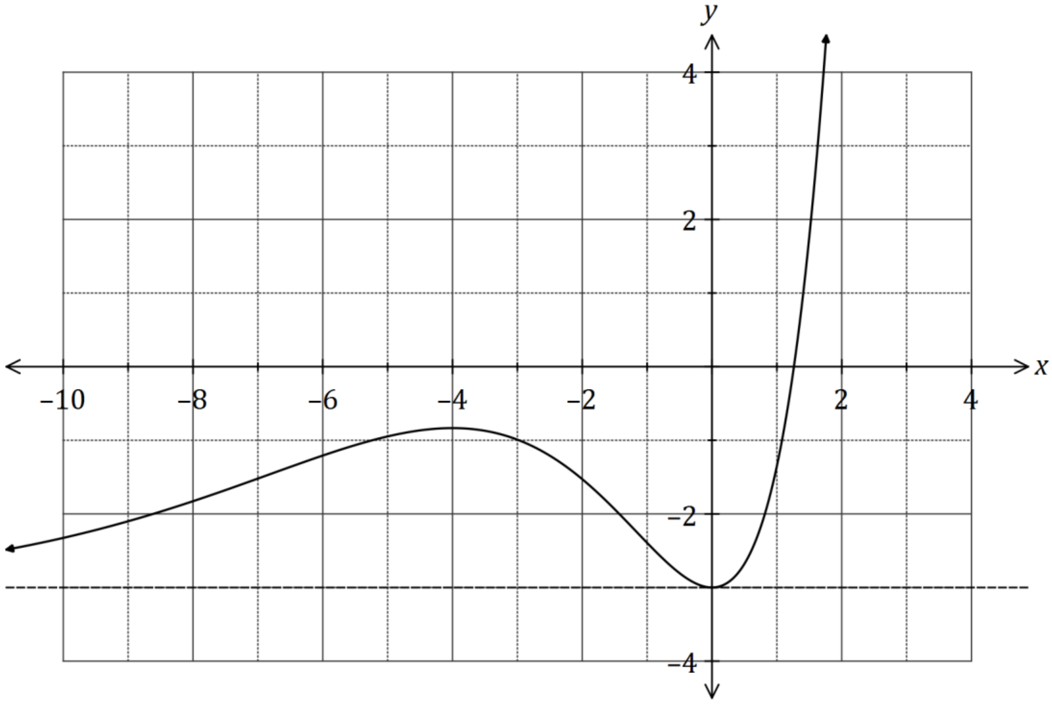
Solution
$\mu = 8.31\text{ h}$ $\dfrac{\sigma}{\sqrt{48}} = 0.05 \Rightarrow \sigma \approx 0.3464\text{ h}$
Specific behaviours
✓ uses sample mean as best estimate for μ ✓ indicates sample sd smaller by factor $\sqrt{48}$ ✓ correct estimate for σ

(b) The batteries in one of the boxes lasted for a total of 396 hours. Use this sample of 48 batteries to construct a 95% confidence interval for the lifetime of this type of AAA battery. (3 marks)

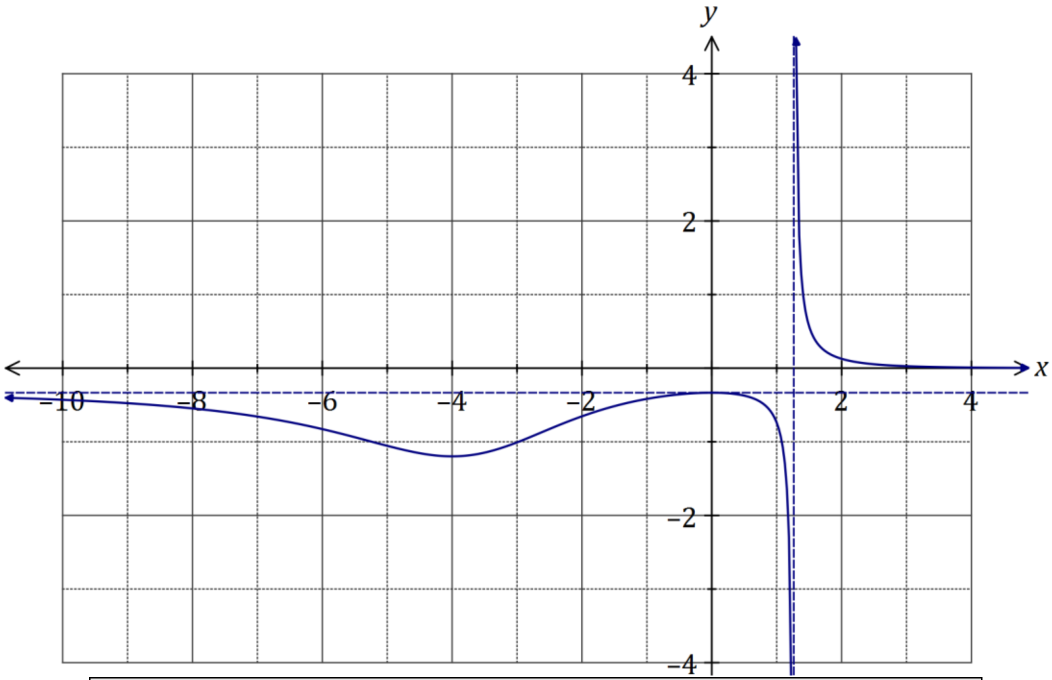
Solution
$\bar{x} = \dfrac{396}{48} = 8.25\text{ h}$ $1.96 \times 0.05 = 0.098$ $8.25 \pm 0.098 = (8.152, 8.348)$
Specific behaviours
✓ calculates mean   ✓ calculates margin of error   ✓ states interval

Question 14 **(8 marks)**

The graph of $y = f(x)$ has asymptote with equation $y = -3$, and is shown below.

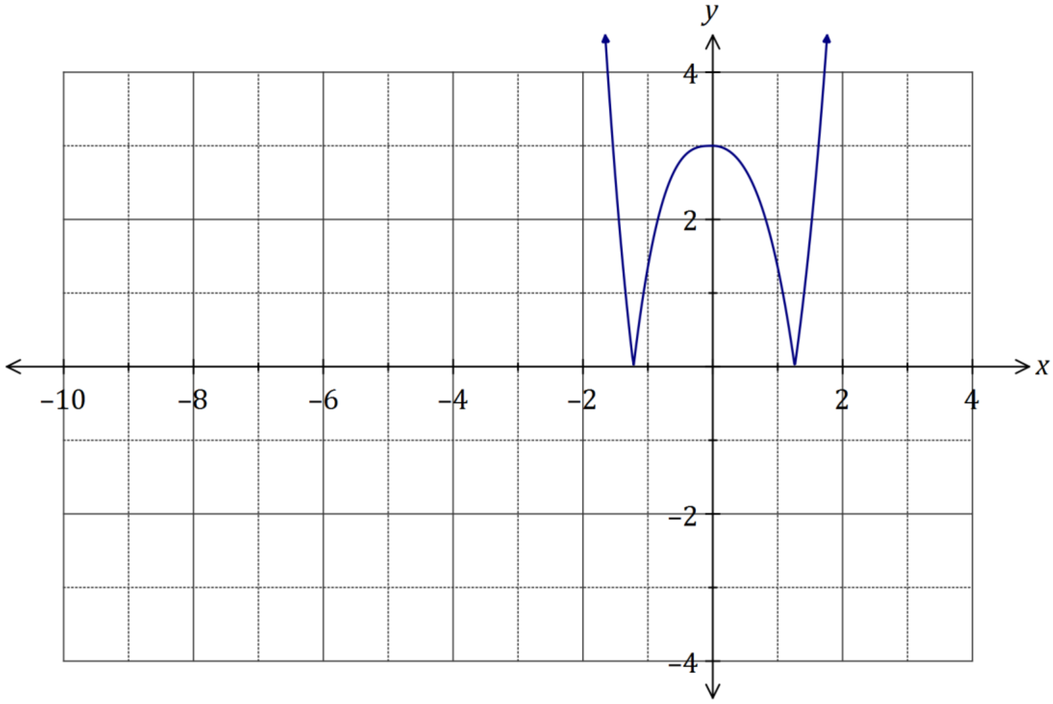


(a) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks)



Solution
See graph
Specific behaviours
✓ indicates vertical asymptote $x \approx 1.2$ ✓ indicates correct curvature and behaviour for $x > 1.2$ ✓ indicates as $x \to -\infty$, approaches horizontal asymptote $y = -\frac{1}{3}$ ✓ indicates local minimum at about $(-4, -1.2)$ ✓ indicates local maximum at $\left(0, -\frac{1}{3}\right)$

(b) Sketch the graph of $y = |f(|x|)|$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ indicates two cusps at $x \approx \pm 1.2$ ✓ indicates symmetry about $x = 0$ ✓ indicates y-intercept at $(0, 3)$

Question 15 **(6 marks)**

(a) $x = A\cos(nt)$ since start at end point

$\therefore \quad x = 15\cos\left(\frac{\pi t}{5}\right)$ ✓✓

and $\frac{dx}{dt} = -3\pi\sin\left(\frac{\pi t}{5}\right)$

$\therefore$ Max speed = 3π m/sec ✓

(b) $\frac{d^2x}{dt^2} = -\frac{3\pi^2}{5}\cos\left(\frac{\pi t}{5}\right)$ ✓

$\therefore$ Min acceleration = $-\frac{3\pi^2}{5}$ m/sec^2 is at $x = 15$ m ✓✓

Question 16 **(8 marks)**

Consider the function $h(x) = \dfrac{4}{x \ln x}$.

(a) Using your calculator, or otherwise, write down the exact area bounded by $y = h(x)$ and the lines $y = 4, x = e$ and $x = e^4$. (2 marks)

Solution
$\int_e^{e^4} 4 - h(x)\,dx = 4e^4 - 4e - 8\ln 2$
Specific behaviours
✓ writes integral ✓ evaluates in exact form

(b) $h(x)$ can be written in the form $f\big(g(x)\big) \cdot g'(x)$. State suitable functions for f and g. (2 marks)

Solution
$f(x) = \dfrac{4}{x}$ and $g(x) = \ln x$ (NB any $g(x)$ of form $a\ln x$)
Specific behaviours
✓ states $f(x)$ ✓ states $g(x)$

(c) Show how to use integration to obtain the answer to (a) without a CAS calculator. (4 marks)

Solution
$\int_e^{e^4} 4 - h(x)\,dx = \int_e^{4} 4\,dx - \int_e^{e^4} h(x)\,dx$ $\int_e^{e^4} 4\,dx = 4e^4 - 4e$ If $u = g(x)$, then $\int_e^{e^4} h(x)\,dx = \int_1^4 \dfrac{4}{u}\,du$ $= [4\ln\lvert u\rvert]_1^4$ $= 4\ln 4$ $= 8\ln 2$ Hence area $= 4e^4 - 4e - 8\ln 2$
Specific behaviours
✓ shows and evaluates integral under $y = 3$ ✓ uses substitution for integral of $h(x)$ ✓ evaluates integral of $h(x)$ ✓ subtracts integrals

Question 17 **(9 marks)**

The serving sizes of peanuts dispensed by a machine have been observed to have a mean of 220 g and a standard deviation of 3.4 g.

(a) A random sample of 75 serves of peanuts are taken from the machine and the serving size measured in each case. Determine the probability that

(i) the sample mean will be no more than 220.2 mL. (3 marks)

Solution
let $\bar{X}$ = sample mean $\bar{X} \sim N\left(220, \frac{3.4^2}{75}\right)$ NB $\sqrt{3.4^2 \div 75} \approx 0.3926$ $P(\bar{X} < 220.2) = 0.6948$
Specific behaviours
✓ indicates sample mean is normal rv ✓ states correct parameters of normal distribution ✓ states probability

(ii) the total weight of peanuts dispensed will be between 16.482 kg and 16.509 kg. (3 marks)

Solution
$\frac{16482}{75} = 219.76, \frac{16509}{75} = 220.12$ $P(219.76 \le \bar{X} \le 220.12) = 0.3496$
Specific behaviours
✓ calculates sample mean serving sizes ✓ writes statement of probability calculated ✓ states probability

(b) After servicing of the machine, an inspector plans to construct a 95% confidence interval for the serving size dispensed by the machine. Determine the sample size they should take so that the width of the interval is no more than 1.5 g, and note any assumptions made. (3 marks)

Solution
$z_{0.95} = 1.96$ $n = \left(\frac{1.96 \times 3.4}{0.75}\right)^2 = 78.9 \Rightarrow n = 79$ servings Assumed: standard deviation is still 3.4 g, approximate normality of sampling distribution.
Specific behaviours
✓ indicates correct z-score and interval half-width ✓ calculates sample size as integer ✓ notes at least one valid assumption

Question 18 **(9 marks)**

The position vector of a particle at time t seconds, $t \geq 0$, is shown below, with distances in cm.

$$\mathbf{r}(t) = \begin{pmatrix} 4\cos t - 1 \\ 3\sin t - 1 \end{pmatrix}$$

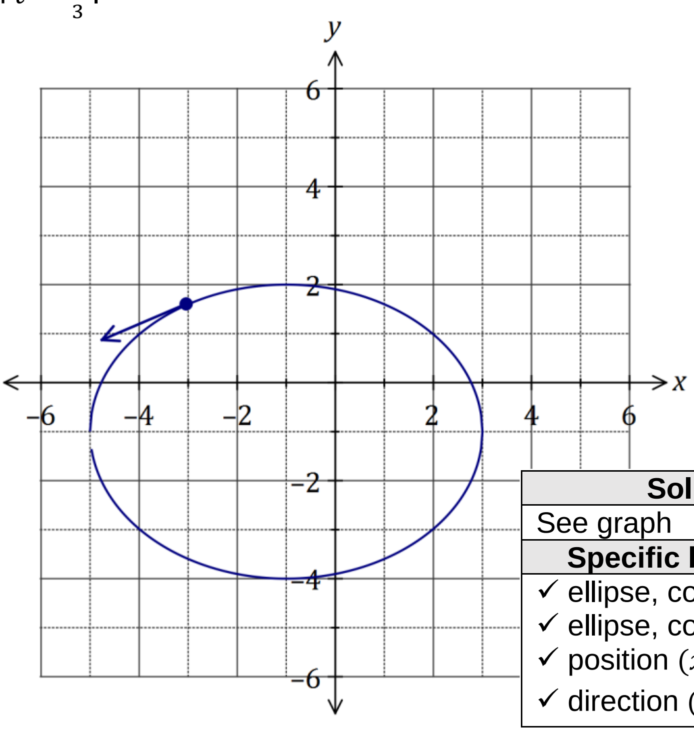
(a) Determine the speed of the particle when $t = \frac{2\pi}{3}$. (3 marks)

Solution
$\mathbf{v}(t) = \begin{pmatrix} -4\sin t \\ 3\cos t \end{pmatrix}, \mathbf{v}\left(\frac{2\pi}{3}\right) = \begin{pmatrix} -2\sqrt{3} \\ -1.5 \end{pmatrix}$, speed $= \frac{\sqrt{57}}{2} \approx 3.77$ cm/s
Specific behaviours
✓ differentiates to obtain velocity vector ✓ substitutes time to obtain velocity ✓ determines magnitude of velocity

(b) Express the path of the particle as a Cartesian equation. (2 marks)

Solution
$\cos t = \frac{x+1}{4}, \sin t = \frac{y+1}{3}$ $\left(\frac{x+1}{4}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$
Specific behaviours
✓ rearranges for $\sin t$ and $\cos t$ ✓ eliminates t to obtain Cartesian equation

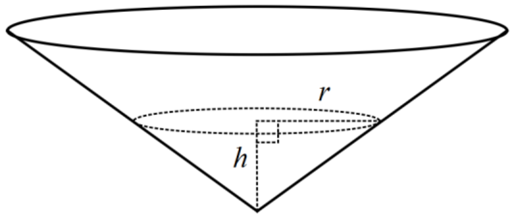
(c) Sketch the path of the particle on the axes below, indicating its position and the direction it is moving when $t = \frac{2\pi}{3}$. (4 marks)



Solution
See graph
Specific behaviours
✓ ellipse, correct centre ✓ ellipse, correct shape ✓ position ($x = -3$) ✓ direction (slope $\approx \frac{1}{2}$)

Question 19 **(12 marks)**

An inverted right cone of diameter 90 cm and height 15 cm is being filled with water at a constant rate of 5π cm^3 per second. Initially the cone contains 56π cm^3 of water. Let r be the radius of the surface of the water and h be the depth of water after t seconds.



(a) Show that the relationship between the volume of water in the cone, V cm^3, and the radius is given by $V = \frac{\pi}{9}r^3$. (2 marks)

Solution
$h = \frac{r}{3}, V = \frac{1}{3}\pi r^2\left(\frac{r}{3}\right) = \frac{\pi}{9}r^3$
Specific behaviours
✓ relates r and h ✓ substitutes and simplifies cone volume

(b) Show that $\frac{dr}{dt} = \frac{15}{r^2}$. (2 marks)

Solution
$\frac{dV}{dr} = \frac{\pi}{3}r^2$ and $\frac{dV}{dt} = 5\pi$ $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{3}{\pi r^2} \times 5\pi = \frac{15}{r^2}$
Specific behaviours
✓ indicates dV/dr and dV/dt ✓ uses chain rule to obtain expression for dr/dt

(c) Determine the rate of change of radius r when $t = 5$. (2 marks)

Solution
$V = 56\pi + 25\pi = 81\pi$ $\frac{\pi}{9}r^3 = 81\pi \Rightarrow r = 9$ $\frac{dr}{dt} = \frac{15}{9^2} = \frac{5}{27}$ cm/s
Specific behaviours
✓ calculates radius ✓ substitutes to obtain rate

(d) Use the differential equation from (b) to determine a relationship between the radius r and time t. (4 marks)

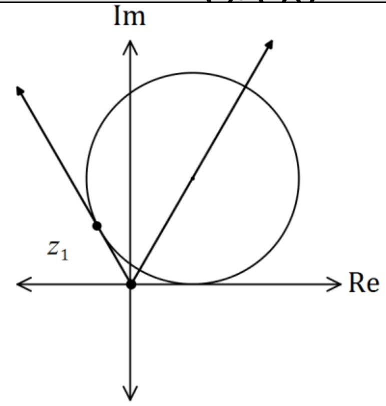
Solution
$\frac{dr}{dt} = \frac{15}{r^2}$ $\int r^2\, dr = \int 15\, dt$ $\frac{r^3}{3} = 15t + c$ $c = \frac{9^3}{3} - 15(5) = 168$ $r^3 = 45t + 504$
Specific behaviours
✓ separates variables   ✓ anti-differentiates correctly   ✓ determines constant   ✓ correct relationship

(e) Calculate the time required to completely fill the cone. (2 marks)

Solution
$45^3 = 45t + 504$ $t = 2013.8\ s$
Specific behaviours
✓ writes equation   ✓ determines time

Question 20 **(7 marks)**

(a) Sketch on an Argand diagram the locus L of the complex number z given by $\arg z = \frac{\pi}{3}$. (1 mark)

Solution (a), (b)(i)

Specific behaviours
✓ (a) line segment with gradient > 1 ✓ (b)(i) centre on L ✓ (b)(i) touching Re axis

(b) A circle C, of radius 9, has its centre lying on L and just touches the line $\text{Im}(z) = 0$.

(i) Draw C on your diagram above. (2 marks)

(ii) Determine the equation of C in the form $|z - z_0| = k$. (2 marks)

Solution
$\lvert z - (3\sqrt{3} + 9i) \rvert = 9$
Specific behaviours
✓ x-coord ✓ correct equation

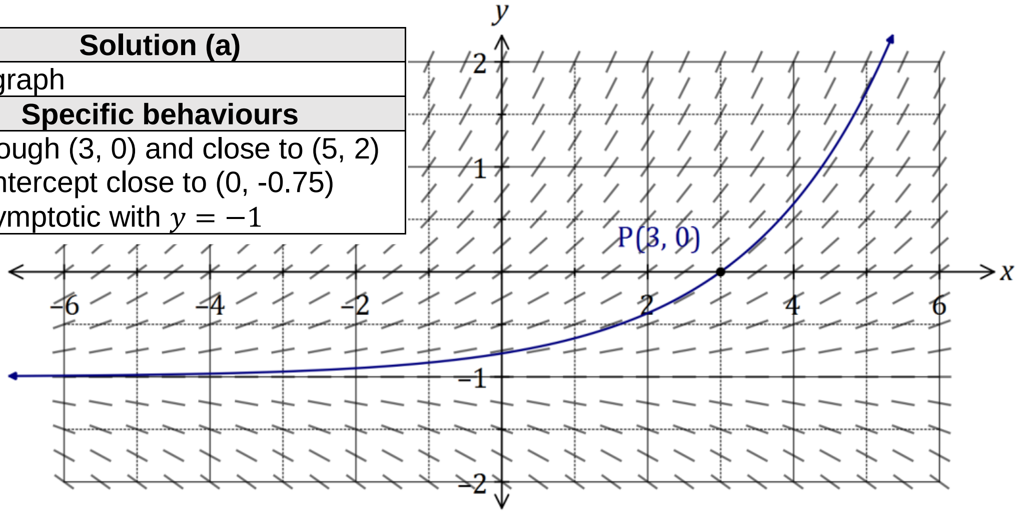
(iii) The complex number z_1 lies on C. Determine the maximum value of $\arg z_1$, where $-\pi < \arg z_1 \leq \pi$. (2 marks)

Solution
Maximum $\arg z_1 = \frac{2\pi}{3}$
Specific behaviours
✓ indicates congruent triangles ✓ correct maximum value

Question 21 **(9 marks)**

A first-order differential equation has a slope field as shown below.

Solution (a)
See graph
Specific behaviours
✓ through (3, 0) and close to (5, 2)
✓ y-intercept close to (0, -0.75)
✓ asymptotic with $y = -1$



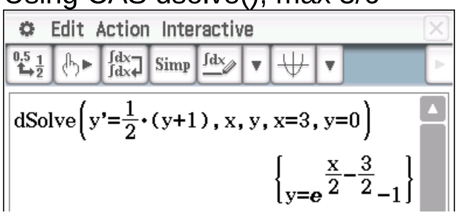
(a) Sketch the solution of the equation that passes through $P(3, 0)$, where the value of the slope is 0.5. (3 marks)

(b) The general differential equation for the slope field is of the form below, where a and b are constants:

$$\frac{dy}{dx} = a(y + b)$$

Derive the solution to this equation that passes through P in the form $y = f(x)$. (6 marks)

Solution
$\frac{dy}{dx} = 0 \text{ when } y = -1 \Rightarrow b = 1$
$0.5 = a(0 + 1) \Rightarrow a = \frac{1}{2}$
$\int \frac{1}{y+1}\,dy = \int \frac{1}{2}\,dx$
$\ln\lvert y + 1\rvert = \frac{x}{2} + c$
$P(3, 0) \Rightarrow c = -\frac{3}{2}$
$y > -1 \Rightarrow \ln(y + 1) = \frac{x}{2} - \frac{3}{2}$
$y = e^{\left(\frac{x}{2} - \frac{3}{2}\right)} - 1$
Specific behaviours
✓ indicates value of b
✓ determines value of a
✓ separates variables and antidifferentiates
✓ determines constant
✓ considers $y > -1$ when rearranging for y
✓ expresses as required

Solution
Using CAS dsolve(), max 3/6

Specific behaviours
✓ indicates value of b
✓ determines value of a
✓ expresses as required

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________